

APPENDIX A

Arithmetic Mean vs. Geometric Mean

When working with the returns to risky assets, it is sometimes helpful to determine their mean or average return. There are two methods to determine the average return to an asset: the arithmetic mean and geometric mean. The arithmetic mean is simply the sum of all of the returns divided by the number of periods over which the sum total is calculated. This is also called the average, or average return. For example, if we are interested in the average mean return for some asset, we have

$$\text{Arithmetic mean} = \frac{1}{T} \sum_{t=1}^T \text{Asset return}(t)$$

where

$\sum_{t=1}^T$ = the summation notation that means that we are adding up all of the observations over T periods
Asset return(t) = the return to the Asset in the t th year
 T = the number of periods over which we calculate the average

Consider the returns for Asset X shown in Exhibit A.1 over a 10-year time horizon. We use this data to demonstrate the difference between the arithmetic and geometric mean.

Using the above equation and the data in the exhibit, the arithmetic mean is calculated as:

$$\begin{aligned} \text{Arithmetic mean} &= [10\% + 14\% + (-7\%) + (-3\%) + 22\% \\ &\quad + 16\% + 8\% + 13\% + 15\% + 12\%]/10 \\ &= 100\%/10 = 10\% \end{aligned}$$

EXHIBIT A.1 The Returns to Asset X

Year	Return
1	10%
2	14
3	-7
4	-3
5	22
6	16
7	8
8	13
9	15
10	12

This is the average annual return produced by risky asset X over the past 10 years.

The arithmetic mean is the simplest calculation to determine the average return. The average return on an asset, observed over a long period of time, is often used as the expected return for future years. Although the future return is unknown and cannot be predicted with great accuracy, the historical average return is as good a guess as any of what the return will be in the future.

The geometric mean is a bit more complicated. It uses compounding to determine the mean return. For a set of observations related to an asset return stream, the geometric mean is equal to

$$1 + R(G) = \sqrt[T]{[1 + R(1)] \times [1 + R(2)] \times \cdots \times [1 + R(T)]}$$

where

- $R(G)$ = the return for the geometric mean
- $R(1), R(2), R(T)$ = the returns to asset X in periods 1, 2, all the way to period T
- T = the number of periods over which we calculate the geometric mean
- $\sqrt[T]{}$ means that we take the Tth root of our compound return stream to determine $R(G)$

Using the data in Exhibit A.1 and the previous equation, we calculate the geometric mean as:

$$1 + R(G) = \sqrt[10]{(1 + 10\%)(1 + 14\%)(1 - 7\%)(1 - 3\%)(1 + 22\%)(1 + 16\%)(1 + 8\%) \times (1 + 13\%)(1 + 15\%)(1 + 12\%)}$$

$$1 + R(G) = 1.0967$$

$$R(G) = 9.67\%$$

We note that the geometric mean is slightly lower than the arithmetic mean. This is the case when the returns are changing through time. In fact, there is an approximate relationship between the two:

$$R(G) \approx R(A) - \frac{1}{2} \text{Var}(R)$$

where $\text{Var}(R)$ is the variance of the rate of return on asset X . It can be seen that if the annual rate of return on the asset has no volatility, then the geometric mean and arithmetic mean will be equal.

One more observation is appropriate. The geometric mean is also used in the calculation for the cumulative average growth rate (CAGR). CAGR assumes that an asset, cash flow, or some other random variable grows at a constant rate of return compounded over a sample period of time. For example, assume that we purchase a stock of company A at \$50 and hold it for three years until it reaches the value of \$100. What is our CAGR? The calculation is simple: We combine our holding period return calculation with the equation for the geometric mean.

First, determine the holding period return as

$$\$100/\$50 = 2$$

Then our CAGR is equal to

$$1 + R(G) = \sqrt[3]{2.0} = \sqrt[3]{(1 + \text{CAGR}) \times (1 + \text{CAGR}) \times (1 + \text{CAGR})}$$

$$1 + R(G) = 1.26 = (1 + \text{CAGR})$$

$$R(G) = 26\% = \text{CAGR}$$